REFERENCES

1. Sedov, L. I., Plane Problems of Hydrodynamics and Aerodynamics. Moscow, "Nauka", 1965.
2. Cherpanov, G. P., The Mechanics of Brittle Fracture. Moscow, "Nauka", 1974.
3. Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity. Moscow, "Nauka", 1966.

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CRACK PROPAGATION AT VARIABLE VELOCITY

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The plane problem of rectilinear crack propagation in an elastic medium subjected to arbitrary variable loads is considered. The position of the crack tip is given as an arbitrary monotonically increasing differentiable function of time such that the velocity of crack propagation at any time is less than the Rayleigh wave velocity. An expression is obtained for the stresses on the crack plane ahead of the tip, particularly the stress intensity factors at its tip.

A fracture criterion permitting determination of the law of crack tip propagation under given external conditions is used to analyze crack propagation in fracture mechanics. In particular, the Griffith energy criterion which can be written as [1]

$$2\gamma(v) = -\frac{\pi}{2\mu b^2 v R(1/v)}\left(\sqrt{v^{-2}-a^{-2}}\,k_1^2 + \sqrt{v^{-2}-b^{-2}}\,k_2^2\right) + \frac{\pi}{2\mu v\sqrt{v^{-2}-b^{-2}}}\,k_3^2 \quad (0.1)$$

$$R(s) = (2s^2 - b^{-2})^2 + 4s^2\sqrt{a^{-2}-s^{-2}}\sqrt{b^{-2}-s^{-2}}$$

can be used for an ideally brittle, linearly elastic medium.

Here μ is the shear modulus, a and b are the longitudinal and transverse wave velocities, v is the velocity of crack propagation, $\gamma(v)$ is the effective surface energy which is considered a characteristic function of the crack propagation rate for a given material, and k_1, k_2, k_3 are the stress intensity factors for the three main modes of fracture: tensile, in-plane shear, and anti-plane shear (longitudinal shear), respectively. The function $R(s)$ vanishes at the points $s = \pm c^{-1}$, where c is the Rayleigh wave velocity.

In order to apply the criterion (0.1) to a specific problem, the stress intensity factors k_i must be known as functionals of the crack tip motion for which the solution of the corresponding dynamic problem of elasticity theory must be obtained for an arbitrary crack tip motion. This has been done in [2] for the particular case of an anti-plane shear crack. This case is simplest since only transverse waves polarized parallel to the crack edge originate. Recently Freund [3] found an expression for the intensity factor for a semi-infinite tensile crack being propagated at piecewise-constant velocity under the effect of static loads by using a clever semi-inverse method. Considering propagation at an arbitrary variable velocity as the limit case of a piecewise-constant velocity, he arrived at the deduction that the expression he obtained is valid even in the general case. The Freund result possesses two disadvantages. Firstly, this result has no foundation.

Indeed, the stress intensity factor is defined as the limit

$$k_i = \lim \sqrt{2\,(x - l\,(t))}\ \sigma_i\,(x, t) \qquad (0.2)$$

where $l\,(t)$ is the coordinate of the crack tip at time t and $\sigma_i\,(x, t)$ are the stress vector components of the continuation of the crack. In order to calculate the intensity factor at the tip of a crack propagating in an arbitrary way, we should first pass to the limit from the piecewise-constant to the variable velocity, and then evaluate the limit (0.2). At the same time Freund first passed to the limit according to (0.2) for the crack being propagated at the piecewise-constant velocity, and only then performed the passage to the limit to the arbitrary propagation velocity. The possibility of changing the order of passing to the limit is not evident and is nowhere given a foundation in [3] (*). However, as will be shown below, the Freund result turns out to be true. The second disadvantage is that Freund succeeded in obtaining the solution only for particular cases of time-independent loads [3], and plane-wave loading of a semi-infinite crack [4], which permits consideration of finite crack propagation even for these particular and most simple loads only for times preceding the time of longitudinal wave arrival from one tip of the crack to its second tip.

1. Formulation of the problem. Let an infinite elastic medium with shear modulus μ and longitudinal and transverse wave propagation velocities a and b, respectively, fill the space outside the crack

$$x_1 = 0, \quad l_-\,(t) < x_2 < l_+\,(t), \quad -\infty < x_3 < \infty \qquad (1.1)$$

Let us assume that all the fractions applied to the medium are independent of the coordinate x_3 (plane problem). We write the equations of motion and Hooke's law as

$$\sigma_{\alpha,\beta\beta} = \rho u_\alpha^{\cdot\cdot}, \quad \sigma_{3\alpha,\alpha} = \rho u_3^{\cdot\cdot}, \quad \rho = \mu b^{-2} \qquad (1.2)$$

$$\sigma_{\alpha\beta} = \mu(\delta_{\alpha\beta}\,(\varkappa^2 - 2)u_{\lambda,\lambda} + u_{\alpha,\beta} + u_{\beta,\alpha})$$

$$\sigma_{3\alpha} = \mu u_{3,\alpha}, \quad \alpha,\ \beta,\ \lambda = 1,\ 2, \quad \varkappa = a\,/\,b$$

The dots denote differentiation with respect to time t here, and the subscript after the comma denotes differentiation with respect to the corresponding spatial coordinate. Here and henceforth, summation is over the repeated Greek subscripts. Summation is not carried out over the Latin subscripts. The effect of body forces is not taken into account in the equations of motion.

If body forces act on the medium and the initial conditions are not homogeneous, then by using the linearity of the equations we can proceed as follows. Let σ_{ik}°, u_i° denote the solution corresponding to the same body forces and initial conditions but for a medium without a crack. The construction of such a solution raises no difficulties in principle. Let us represent the solution of the problem for a medium with a crack as the sum: $\sigma_{ik} + \sigma_{ik}^\circ$, $u_k + u_k^\circ$. Then u_k, σ_{ik} will satisfy the equations of motion in the absence of the mass forces (1.2), and homogeneous initial conditions

$$u_k = u_k^{\cdot} = 0 \quad \text{for} \quad t \leqslant 0 \qquad (1.3)$$

*) Freund [4] recently examined the case of plane stress wave incidence on a semi-infinite crack being propagated, by the same method. The remarks made here remain valid also relative to [4].

Let the stress resultants $-p_i^\circ(x_2, t)$ be given on the crack; then the boundary conditions for σ_{ik} are

$$\sigma_i(x_2, t) \equiv \sigma_{i1}(0, x_2, t) = -p_i(x_2, t) \qquad (1.4)$$

$$\text{for} \quad x_1 = 0, \; l_-(t) < x_2 < l_+(t)$$

where

$$p_i(x_2, t) = p_i^\circ(x_2, t) + \sigma_{i1}^\circ(x_2, 0, t)$$

With respect to the functions $l_-(t)$, $l_+(t)$, governing the crack propagation laws, we assume

$$-c < l_-^{\boldsymbol\cdot}(t) \leqslant 0, \qquad 0 \leqslant l_+^{\boldsymbol\cdot}(t) < c \qquad (1.5)$$

These conditions can be weakened in the case of a shear crack when all the stress resultants applied to the medium are parallel to the x_3-axis (i.e. the crack tip) by replacing c in (1.5) by b.

Let us solve (1.2) by using a double Laplace transform in the coordinate x_2 and time t. Let us denote the transforms of the functions by the same letters as the originals but making the distinction explicit (where necessary) by writing the arguments

$$u_i(x_1, q, p) = \int_0^\infty e^{-pt} \int_{-\infty}^\infty e^{-qx_2} u_i(x_1, x_2, t)\, dx_2\, dt \qquad (1.6)$$

$$\sigma_{ik}(x_1, q, p) = \int_0^\infty e^{-pt} \int_{-\infty}^\infty e^{-qx_2} \sigma_{ik}(x_1, x_2, t)\, dx_2\, dt$$

For brevity, by $\sigma_i(q, p)$ we denote the Laplace transform of the boundary values of the stress vector components on the x_2-axis: $\sigma_i(x_2, t) = \sigma_{i1}(0, x_2, t)$.

Equations (1.2) are solved in a standard manner, and the solution is

$$u_1(x_1, q, p) = \frac{1}{\mu R(q/p)\, p}\Big(e^{-|x_1| p r_a}\Big(2\frac{q}{p} r_a r_b \sigma_2(q, p) - \qquad (1.7)$$

$$\Big(b^{-2} - \frac{2q^2}{p^2}\Big) r_a \sigma_1(q, p) \operatorname{sgn} x_1\Big) -$$

$$\frac{q}{p} e^{-|x_1| p r_b}\Big(\Big(b^2 - \frac{2q^2}{p^2}\Big)\sigma_2(q, p) + 2\frac{q}{p} r_a \sigma_1(q, p) \operatorname{sgn} x_1\Big)\Big)$$

$$u_2(x_1, q, p) = \frac{1}{\mu R(q/p)\, p}\Big(\frac{q}{p} e^{-|x_1| p r_a}\Big(\Big(b^2 - \frac{2q^2}{p^2}\Big)\sigma_1(q, p) -$$

$$2\frac{q}{p} r_b \sigma_2(q, p) \operatorname{sgn} x_1\Big) - e^{-|x_1| p r_b}\Big(2\frac{q}{p} r_a r_b \sigma_1(q, p) -$$

$$\Big(b^2 - \frac{2q^2}{p^2}\Big) r_b \sigma_2(q, p) \operatorname{sgn} x_1\Big)\Big)$$

$$u_3(x_1, q, p) = -\frac{1}{\mu p r_b} e^{-|x_1| p r_b} \sigma_3(q, p) \operatorname{sgn} x_1$$

$$r_a = \sqrt{a^{-2} - \frac{q^2}{p^2}}, \qquad r_b = \sqrt{b^{-2} - \frac{q^2}{p^2}}$$

Let $w_i(x_2, t)$ denote the discontinuity in the displacement across the x_2-axis

$$w_i(x_2, t) = u_i(+0, x_2, t) - u_i(-0, x_2, t)$$

and $w_i(q, p)$ the corresponding Laplace transform. We then obtain from (1.7)

$$\sigma_i(q, p) + pK_{(i)}(q/p) w_i(q, p) = 0 \qquad (1.8)$$

$$K_{(1)}(s) = \frac{\mu b^2 R(s)}{2\sqrt{a^{-2} - s^2}}, \quad K_{(2)}(s) = \frac{\mu b^2 R(s)}{2\sqrt{b^{-2} - s^2}}, \quad K_{(3)}(s) = {}^1/_2 \mu \sqrt{b^{-2} - s^2} \qquad (1.9)$$

The expressions (1.7) would yield the solution of the problem if the functions $\sigma_i(q, p)$, could be calculated from the boundary conditions, i.e. if the stresses were known on the whole x_2-axis. However, conditions (1.4) yield values of the stresses only on the crack surfaces. But then the displacements should be continuous outside the crack, i.e.

$$w_i(x_2, t) = 0 \quad \text{for} \quad x_2 < l_-(t),\ l_+(t) < x_2 \tag{1.10}$$

The problem would therefore reduce to seeking the stresses $\sigma_i(x_2, t)$ on the continuation of the crack from conditions (1.4) and (1.10) and the functional equation (1.8).

The functions $K_{(i)}(s)$ are encountered in solving problems of elastic wave diffraction by a free half-plane, a crack. It can be shown that the representation

$$R(s) = (b^{-2} - a^{-2})(c^{-2} - s^{-2})S(s)S(-s) \tag{1.11}$$

$$S(s) = \exp\left(-\frac{1}{\pi}\int_{a^{-2}}^{b^{-2}} \operatorname{arctg} \frac{4\xi^2\sqrt{\xi^2 - a^{-2}}\sqrt{b^{-2} - \xi^2}}{(2\xi^2 - b^{-2})^2}\frac{d\xi}{\xi + s}\right)$$

is valid for the function $R(s)$.

The function $S(s)$ is regular in the complex s plane slit along a segment of the real axis between the points $s = -a^{-1}$ and $s = -b^{-1}$, and tends to unity as $s \to \infty$. The expansion (1.11) is the basis of the solution of (1.8) by the Wiener-Hopf method for a fixed or semi-infinite crack being propagated at constant velocity. Although the Wiener-Hopf method is not directly applicable to the problem of nonuniform crack propagation, the expansion (1.11) permits construction of the solution even in this case by reducing the problem to the solution of an integral equation encountered in supersonic flow theory and used in [2] to solve the problem of the anti-plane shear crack.

2. Solution for a semi-infinite crack. Let us examine the particular case of a semi-infinite crack when $l_-(t) = -\infty$. In this case, we seek $l(t)$ instead of $l_+(t)$. Moreover, for brevity let us omit the subscript on the coordinate x_2. Then the boundary conditions (1.4) and (1.10) become

$$\begin{aligned} \sigma_i(x, t) &= -p_i(x, t) \quad &\text{for } -\infty < x < l(t) \\ w_i(x, t) &= 0 &\text{for } l(t) < x < \infty \end{aligned} \tag{2.1}$$

It is necessary to find the solution of (1.8) under the conditions (2.1) such that

$$\sigma_i(x, t) = \frac{k_i(t)}{\sqrt{2(x - l(t))}} + O(1) \quad \text{for} \quad x \to l(t) + 0$$

We introduce the new functions $F_i(x, t)$, $G_i(x, t)$, defined as follows:

$$\begin{aligned} F_\alpha(q, p) &= \frac{\sqrt{a^{-1} + q/p}\sqrt{b^{-1} + q/p}}{c^{-1} + q/p} S^{-1}\left(\frac{q}{p}\right)\sigma_\alpha(q, p), \quad \alpha = 1, 2 \\ F_3(q, p) &= \sigma_3(q, p) \\ G_\alpha(q, p) &= \frac{\mu(1 - \varkappa^{-2})(c^{-1} - q/p)}{4\sqrt{a^{-1} - q/p}\sqrt{b^{-1} - q/p}} S\left(\frac{-q}{p}\right) w_\alpha(q, p), \quad \alpha = 1, 2 \\ G_3(q, p) &= {}^1/_2\mu w_3(q, p) \end{aligned} \tag{2.2}$$

Then (1.8) becomes

$$\frac{1}{p\sqrt{v_i^{-2}-q^2/p^2}}F_i(q,p)+G_i(q,p)\equiv 0,\ v_1=v_3=b,\ v_2=a \qquad (2.3)$$

In the physical x and t variables,the relationships (2.2) are

$$F_i(x,t)=\mathbf{A}_i^+\sigma_i\equiv\sigma_i(x,t)- \qquad (2.4)$$

$$(1-\delta_{i3})\frac{\partial}{\partial t}\Big[D(-c^{-1})\sqrt{c^{-1}-a^{-1}}\sqrt{c^{-1}-b^{-1}}\int_0^{ct}\sigma_i(x-\eta,t-$$

$$c^{-1}\eta)\,d\eta+\frac{1}{2\pi}\int_{a^{-1}}^{b^{-1}}\{D(-s)\}\frac{\sqrt{s-a^{-1}}\sqrt{b^{-1}-s}}{c^{-1}-s}\int_0^{t/s}\sigma_i(x-\eta,t-s\eta)\,d\eta ds\Big]$$

$$D(s)=(S(s))^{-1},\ \{D(s)\}=D(s+i0)+D(s-i0)$$

$$G_i(x,t)=\mathbf{B}_i^+w_i\equiv C_i\Big[w_i(x,t)+$$

$$(1-\delta_{i3})\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{a^{-1}}^{b^{-1}}\{S(-s)\}\frac{(c^{-1}-s)}{\sqrt{s-a^{-1}}\sqrt{b^{-1}-s}}\int_0^{t/s}w_i(x+\eta,t-s\eta)\,d\eta\,ds\Big]$$

$$C_\alpha={}^1/_4\mu(1-\varkappa^{-2}),\quad \alpha=1,2,\quad C_3={}^1/_2\mu$$

The transforms (2.4) possess the remarkable property that for $x<l(t)$ (i.e. on the crack) $F_i(x,t)$ are calculated in terms of values of $\sigma_i(x',t')$, where $x'<l(t)$ (let us recall that $l(t)<c$) by assumption). In exactly the same way, values of $G_i(x,t)$ on the continuation of the crack are evaluated in terms of values of w_i on the continuation of the crack. Hence, we obtain in place of conditions (2.1)

$$F_i(x,t)=-f_i(x,t)\qquad \text{for } x<l(t) \qquad (2.5)$$
$$G_i(x,t)=0\qquad \text{for } x>l(t)$$

Here $f_i(x,t)$ is a known function related to $p_i(x,t)$ by the transform (2.4)

$$f_i(x,t)=\mathbf{A}_i^+p_i \qquad (2.6)$$

The convolution type transforms (2.4) are invertible. The inverse transforms are

$$\sigma_i(x,t)=(\mathbf{A}_i^+)^{-1}F_i\equiv F_i(x,t)+(1-\delta_{i3})\times \qquad (2.7)$$

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{a^{-1}}^{b^{-1}}\{S(-s)\}\frac{c^{-1}-s}{\sqrt{s-a^{-1}}\sqrt{b^{-1}-s}}\int_0^{t/s}F_i(x-\eta,t-s\eta)\,d\eta\,ds$$

$$w_i(x,t)=(\mathbf{B}_i^+)^{-1}G_i=C_i^{-1}\Big(G_i(x,t)-(1-\delta_{i3})\times$$

$$\frac{\partial}{\partial t}\Big[D(-c^{-1})\sqrt{c^{-1}-a^{-1}}\sqrt{c^{-1}-b^{-1}}\int_0^{ct}G_i(x+\eta,t-c^{-1}-\eta)\,d\eta+$$

$$\frac{1}{2\pi}\int_{a^{-1}}^{b^{-1}}\{D(-s)\}\frac{\sqrt{s-a^{-1}}\sqrt{b^{-1}-s}}{c^{-1}-s}\int_0^{t/s}G_i(x+\eta,t-s\eta)\,d\eta\,ds\Big]\Big)$$

The problem is therefore reduced to finding the function $F_i(x,t)$ from (2.3) and the boundary comditions (2.5). The asymptotic behavior of $F_i(x, t)$ as $x \to l(t)$ is analogous to the behavior of $\sigma_i(x, t)$

$$F_i(x, t) = \frac{m_i(t)}{\sqrt{2(x - l(t)}} + O(1) \quad \text{for} \quad x \to l(t) + 0$$

The intensity factors $m_i(t)$ are here evaluated in terms of $k_i(t)$ by using (2.4) in the form

$$m_i(t) = k_i(t)\left(\delta_{i3} + (1 - \delta_{i3}) D\left(\frac{-1}{v(t)}\right) \frac{\sqrt{1 - v(t)/a}\,\sqrt{1 - v(t)/b}}{1 - v(t)/c}\right) \tag{2.8}$$

where $v(t) = l^{\cdot}(t)$ is the crack propagation velocity at time t.

Let us write Eq. (2.3) in physical variables by taking the inverse Laplace transform. In particular, because of (2.5) we have for $x_0 > l(t_0)$

$$\frac{1}{\pi} \iint\limits_{\Delta_i} \frac{F_i(x, t)\, dx\, dt}{\sqrt{v_i^2 (t_0 - t)^2 - (x_0 - x)^2}} = 0 \tag{2.9}$$

(Δ_i is the triangle $v_i^2(t_0 - t)^2 - (x_0 - x)^2 \geqslant 0,\ 0 \leqslant t \leqslant t_0$).

This equation agrees exactly with (2.5) in [2]. The method of solving it is no different from that used in [2]. Using the fact that for $x_0 > v_i t_0 + l(0)$ the crack is not incident in the domain Δ_i, it can be proved that $F_i(x, t) \equiv 0$ for $x > v_i t + l(0)$. Furthermore, let us introduce the characteristic variables

$$\xi = (v_i t - x) / \sqrt{2}, \qquad \eta = (v_i t + x) / \sqrt{2}$$

and let $\eta^*(\xi)$ denote the coordinate of the crack edge in the characteristic variables, i.e. the solution of the equation

$$\eta^* - \xi = \sqrt{2}\, l\left(\frac{\eta^* + \xi}{v_i \sqrt{2}}\right)$$

Then we write Eq. (2.9) as

$$\frac{1}{\pi} \int\limits_{-l(0)/\sqrt{2}}^{\xi_0} \frac{d\xi}{\sqrt{\xi_0 - \xi}} \int\limits_{-\xi}^{\eta_0} \frac{F_i(\xi, \eta)\, d\eta}{\sqrt{\eta_0 - \eta}} = 0 \quad \text{for} \quad \eta_0 > \eta^*(\xi_0)$$

Or inverting the Abel operator with respect to ξ

$$\int\limits_{-\xi}^{\eta_0} \frac{F_i(\xi, \eta)\, d\eta}{\sqrt{\eta_0 - \eta}} = 0 \quad \text{for} \quad \eta_0 > \eta^*(\xi)$$

Since $F_i(\xi, \eta)$ are known for $\eta < \eta^*(\xi)$, by virtue of (2.5), we rewrite this equation as

$$\int\limits_{\eta^*(\xi)}^{\eta_0} \frac{F_i(\xi, \eta)\, d\eta}{\sqrt{\eta_0 - \eta}} = \int\limits_{-\xi}^{\eta^*(\xi)} \frac{f_i(\xi, \eta)\, d\eta}{\sqrt{\eta_0 - \eta}}$$

The solution of this Abel integral equation in the physical x and t variables reduces to

$$F_i(x_0, t_0) = C_i^+ f_i \equiv \frac{1}{\pi \sqrt{x_0 - l(t^*)}} \int\limits_{x_0 - v_i t_0}^{l(t^*)} f_i\left(x, t_0 \frac{x_0 - x}{v_i}\right) \frac{\sqrt{l(t^*) - x}}{x_0 - x}\, dx \tag{2.10}$$

where t^* is the solution of the equation

$$v_i t_0 - x_0 = v_i t^* - l(t^*)$$

Now, Eqs. (2.10), (2.6) and (2.7) yield the solution of the problem for a semi-infinite crack. In particular, we obtain from (2.10)

$$m_i(t_0) = \frac{\sqrt{2(1 - v(t_0)/v_i)}}{\pi} \int\limits_{l(t_0)-v_i t_0}^{l(t_0)} f_i\left(x, t_0 - \frac{l(t_0) - x}{v_i}\right) \frac{dx}{\sqrt{l(t_0) - x}}$$

Or by virtue of (2.8)

$$k_i(t) = \frac{\sqrt{2}}{\pi}\left[\delta_{i3}\sqrt{\frac{1 - v(t)}{v_3}} + \right. \tag{2.11}$$

$$\left.(1 - \delta_{i3}) S\left(-\frac{1}{v(t)}\right) \frac{(1 - v(t)/c)\sqrt{1 - v(t)/v_i}}{\sqrt{1 - v(t)/a}\sqrt{1 - v(t)/b}}\right] \int\limits_0^{v_i t} f_i\left(l(t) - x, t - \frac{x}{v_i}\right) \frac{dx}{\sqrt{x}}$$

Evidently p_3 can be written in place of f_3 in the expression for k_3, since these functions agree because of (2.6).

Now, in particular, the Freund results can be given a foundation. The expression in the parentheses in (2.11) depends only on the crack propagation velocity $v(t)$ and becomes one for $v(t) = 0$. At the same time the quantities

$$k_i^\circ(l, t) = \frac{\sqrt{2}}{\pi} \int\limits_0^{v_i t} f_i\left(l - x, t - \frac{x}{v_i}\right) \frac{dx}{\sqrt{x}}$$

depend on the position of the crack tip l as on a parameter and are, as is easily surmised, the stress intensity factors calculated for the same loads but for a fixed crack, whose edge is at the point $x = l$ from the very beginning. Therefore, (2.11) can be rewritten as

$$k_i(t) = K_i(v(t))\, k^\circ_i(l(t), t) \tag{2.12}$$

$$K_i(v) = \delta_{i3}\sqrt{1 - \frac{v(t)}{b}} + (1 - \delta_{i3}) S\left(-\frac{1}{v(t)}\right) \frac{(1 - v(t)/c)\sqrt{1 - v(t)/v_i}}{\sqrt{1 - v(t)/a}\sqrt{1 - v(t)/b}}$$

The expression (2.12) agrees formally with the results Freund obtained for the particular cases of a static load [3] and plane wave incidence on the crack [4], and generalizes his results to the case of a semi-infinite crack subjected to arbitrary time-dependent loads.

3. Solution for a finite crack. The solution (2.10) is also valid for a crack of finite length up to the time when the waves from the left tip of the crack reach the right tip, i.e. for t less than t_1^+, where t_1^+ is the solution of the equation

$$l_+(t_1^+) - at_1^+ = l_-(0) \tag{3.1}$$

Under this condition, the integration in (2.6) and (2.10) (where it is everywhere necessary, certainly, to write $l_+(t)$) in place of $l(t)$) is over the crack surface. In the general case, the stresses are expressed for $x > l_+(t)$ in a form analogous to (2.7)

$$\sigma_i(x, t) = (A_i^+)^{-1} F_i^+(x, t) \tag{3.2}$$

The values of F_i^+ are expressed for $x > l_+(t)$ in terms of the values for $x < l_+(t)$ by Eq. (2.10)

$$F_i^+(x_0, t_0) = C_i^+ F_i^+ \tag{3.3}$$

where we write $l_+(t)$, t_+^* in place of $l(t)$, t^*, and the values of F_i^+ are expressed for $x < l_+(t)$ in terms of values of $\sigma_i(x, t)$ for $x < l_+(t)$ by the relationship (2.4)

$$F_i^+(x, t) = \mathbf{A}_i^+ \sigma_i(x, t) \tag{3.4}$$

By repeating the discussion of Sect. 2, we analogously obtain that the stresses for

$x < l_-(t)$ are expressed as

$$\sigma_i(x, t) = (\mathbf{A}_i^-)^{-1}F_i^- \equiv F_i^-(x, t) + (1 - \delta_{i3}) \times \tag{3.5}$$
$$\frac{1}{2\pi}\frac{\partial}{\partial t}\int_{a^{-1}}^{b^{-1}} \{S(-s)\} \frac{c^{-1} - s}{\sqrt{s - a^{-1}}\sqrt{b^{-1} - s}} \int_0^{t/s} F_i^-(x + \eta, t - s\eta)\, d\eta ds$$

The values of $F_i^-(x, t)$ for $x < l_-(t)$ are expressed in terms of the values for $x > l_-(t)$ by the equation

$$F_i^-(x_0, t_0) = \mathbf{C}_i^- F_i^- \equiv \tag{3.6}$$
$$-\frac{1}{\pi\sqrt{l_-(t_-^*) - x_0}} \int_{x_0 + v_i t_0}^{l_-(t_-^*)} F_i^-\left(x, t_0 + \frac{x_0 - x}{v_i}\right) \frac{\sqrt{x - l_-(t_-^*)}}{x_0 - x}\, dx$$

where t_-^* is the solution of the equation

$$v_i t_0 + x_0 = v_i t_-^* + l_-(t_-^*) \tag{3.7}$$

and the values of $F_i^-(x, t)$ for $x > l_-(t)$ are expressed in terms of $\sigma_i(x, t)$ for $x > l_-(t)$ by a relationship analogous to (2.4)

$$F_i^-(x, t) = \mathbf{A}_i^- \sigma_i \equiv \sigma_i(x, t) - \tag{3.8}$$
$$(1 - \delta_{i3})\frac{\partial}{\partial t}\left[D(-c^{-1})\sqrt{c^{-1} - a^{-1}}\sqrt{c^{-1} - b^{-1}} \int_0^{ct} \sigma_i(x + \eta, t - \right.$$
$$\left. c^{-1}\eta)\, d\eta + \frac{1}{2\pi}\int_{a^{-1}}^{b^{-1}} \{D(-s)\} \frac{\sqrt{s - a^{-1}}\sqrt{b^{-1} - s}}{c^{-1} - s} \int_0^{t/s} \sigma_i(x + \eta, t - s\eta)\, d\eta\, ds\right]$$

For $t < t_1^-$, where t_1^- is the solution of the equation

$$l_-(t_1^-) + at_1^- = l_+(0) \tag{3.9}$$

the integration in (3.6) and (3.8) is over the crack surface, where the $\sigma_i(x, t)$ are given and equal $-p_i(x, t)$, so that the relationships (3.5)–(3.8) yield values of $\sigma_i(x, t)$ directly for $x < l_-(t)$. If these values have been calculated, then (3.2)–(3.4) permit evaluation of $F_i^+(x, t)$ and $\sigma_i(x, t)$ in the time range $t_1^+ < t < t_2^+$, where t_2^+ is the solution of the equation

$$l_+(t_2^+) - a(t_2^+ - t_1^-) = l_-(t_1^-) \tag{3.10}$$

Analogically, when $x > l_+(t)$, for calculating $\sigma_i(x, t)$ for $t < t_1^+$, Eqs. (3.5), (3.6) allow us to calculate $\sigma_i(x, t)$ for $x < l_-(t)$ and $t_1^- < t < t_2^-$, where t_2^- is the solution of the equation

$$l_-(t_2^-) + a(t_2^- - t_1^+) = l_+(t_1^+) \tag{3.11}$$

In general, by using (3.2)–(3.4) and (3.5)–(3.8) alternately, the stresses outside the crack can be calculated in terms of the load on its surface in a finite number of steps. This procedure for calculating the stresses on the crack continuation corresponds to multiple wave diffraction by the crack edges.

The stress intensity factors on the crack edges are expressed in a form analogous to (2.11)

$$k_i^{\pm}(t) = \frac{\sqrt{2}}{\pi}\left[\delta_{i3}\sqrt{1 \mp \frac{l_2^{\cdot}(t)}{b}} + (1 - \delta_{i3}) S\left(\mp \frac{1}{l_{\pm}^{\cdot}}(t)\right) \times \right. \tag{3.12}$$

$$\frac{(1 \mp l_\pm^{\cdot}(t)/c)\sqrt{1 \mp l_\pm^{\cdot}(t)/v_i}}{\sqrt{1 \mp l_\pm^{\cdot}(t)/a}\sqrt{1 \mp l_\pm^{\cdot}(t)/b}} \int_0^{v_i t} F_i^\pm\left(l_\pm(t) \mp x, t - \frac{x}{v_i}\right)\frac{dx}{\sqrt{x}}\Bigg]$$

Let us determine the sequence of times $t_k^\pm$ by the recursion relations

$$t_0^\pm = 0, \quad l_\pm(t_k^\pm) - l_\mp(t_{k-1}^\mp) = \pm\, a(t_k^\pm - t_{k-1}^\mp) \qquad (3.13)$$

Then if the crack tip motion $l_\pm(t)$ has already been calculated for $t < t_{k-1}^\pm$, then the functions $l_\pm(t)$ are found for $t_{k-1}^\pm < t \leqslant t_k^\pm$ as the solution of differential equations obtained by substituting the values (3.1) for the stress intensity factor in the fracture condition (0.1), where we should put $v = \pm\, l_\pm^{\cdot}(t)$.

Therefore, the relationships (0.1), (3.2), (3.4)–(3.8) and (3.12) permit the complete solution to be obtained for the problem of crack propagation for any time. Knowing $\sigma_i(x, t)$, the displacements can then be calculated at an arbitrary point of the medium by means of (1.7).

The numerical realization of this solution and obtaining physical deductions for a specific kind of load will be a very complex problem. Indeed, even for $t < t_1^\pm$ the evaluation of quintuple integrals is required to obtain the stresses $\sigma_i(x, t)$ outside the crack, and triple integrals to obtain the intensity factors. It is apparently more convenient to use difference methods in place of this analytical solution in order to obtain the stresses. However, these difference methods possess the disadvantage that they do not permit obtaining the stress intensity factors directly, and therefore, studying the laws of crack motion. Use of combination of a difference method to compute the stresses and the analytical expression (3.12) for the stress intensity factors would probably have the best prospects.

In conclusion, let us note that the solution simplifies for $i = 3$ and reduces to that obtained in [2].

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REFERENCES

1. Kostrov, B. V., Nikitin, L. V. and Flitman, L. M., Mechanics of Brittle Fracture. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, №3, 1969.

2. Kostrov, B. V., Unsteady propagation of longitudinal shear cracks. PMM Vol. 30, №6, 1966.

3. Freund, L. B., Crack propagation in an elastic solid subjected to general loading, II. Nonuniform rate of extension. J. Mech. and Phys. Solids, Vol. 20, №3, 1972.

4. Freund, L. B., Crack propagation in elastic solids subjected to general loading, III. Stress wave loading. J. Mech. and Phys. Solids, Vol. 21, №2, 1973.

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